

* Green's theorem :-

The theorem states that "If ϕ and ψ are two continuously differentiable scalar function such that $\nabla\phi$ and $\nabla\psi$ are also continuously differentiable, then

$$\iiint_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) dV = \iint_S (\phi \nabla \psi - \psi \nabla \phi) \cdot d\vec{S}$$

Proof:

Let a vector function \vec{V} be given by \vec{V}

$$\vec{V} = \phi \nabla \psi \quad (1)$$

where ϕ and ψ are scalar function

so that \vec{V} is product of a scalar function ϕ and the gradient of another scalar ψ .

Writing equation (1) in terms of the components we get

$$V_x = \phi \frac{\partial \psi}{\partial x}, \quad V_y = \phi \frac{\partial \psi}{\partial y}, \quad V_z = \phi \frac{\partial \psi}{\partial z}$$

$$\therefore \nabla \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$= \frac{\partial}{\partial x} \left(\phi \frac{\partial \psi}{\partial x} \right) + \frac{\partial}{\partial y} \left(\phi \frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial z} \left(\phi \frac{\partial \psi}{\partial z} \right)$$

$$= \left\{ \phi \frac{\partial^2 \psi}{\partial x^2} + \frac{d\phi}{dx} \frac{\partial \psi}{\partial x} \right\} + \left\{ \phi \frac{\partial^2 \psi}{\partial y^2} + \frac{d\phi}{dy} \frac{\partial \psi}{\partial y} \right\} +$$

$$\left\{ \phi \frac{\partial^2 \psi}{\partial z^2} + \frac{\partial \phi}{\partial z} \frac{\partial \psi}{\partial z} \right\} \\
 = \left\{ \phi \frac{\partial^2 \psi}{\partial x^2} + \phi \frac{\partial^2 \psi}{\partial y^2} + \phi \frac{\partial^2 \psi}{\partial z^2} \right\} + \left\{ \frac{\partial \phi}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial \psi}{\partial y} + \frac{\partial \phi}{\partial z} \frac{\partial \psi}{\partial z} \right\} \\
 = \phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi \quad (2)$$

By Gauss's theorem of divergence, we have

$$\iint_S \vec{v} \cdot \vec{n} \, ds = \iiint_V (\nabla \cdot \vec{v}) \, dv \quad (3)$$

Hence substituting from equations (1), (2), (3) we get

$$\iint_S (\psi \nabla \phi) \cdot \vec{n} \, ds = \iiint_V (\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi) \, dv \quad (4)$$

If we interchange ϕ and ψ in equation (4), we get

$$\iint_S (\psi \nabla \phi) \cdot \vec{n} \, ds = \iiint_V (\psi \nabla^2 \phi + \nabla \psi \cdot \nabla \phi) \, dv \quad (5)$$

Now subtracting equation (5) from (4), we get

$$\iint_S (\phi \nabla \psi - \psi \nabla \phi) \cdot \vec{n} \, ds = \iiint_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \, dv$$

$$\text{or } \iint_S (\phi \nabla \psi - \psi \nabla \phi) \cdot \vec{n} \, dS = \iiint_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \, dV \quad (6)$$

This is Green's theorem.

Now we derive Green's Theorem from Green's Theorem:

If we put $\phi = 1$, $\psi = \psi$ in equation (6), we have

$$\iiint_V \Delta^2 \psi \, dV = \iint_S (\text{grad } \psi) \cdot d\vec{s} \quad \text{or} \quad \iiint_V \text{div}(\text{grad } \psi) \, dV = \iint_S (\text{grad } \psi) \cdot d\vec{s}$$

If we have a vector field $v = \text{grad } \psi$, then this equation leads to

$$\iiint_V \text{div } v \, dV = \iint_S v \cdot d\vec{s} = \iint_S v \cdot \vec{n} \, dS.$$

It is Gauss's theorem.